Mean field effects in a trapped classical gas

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In this article, we investigate mean field effects for a bosonic gas harmonically trapped above the transition temperature in the collisionless regime. We point out that those effects can play also a role in low dimensional system. Our treatment relies on the Boltzmann equation with the inclusion of the mean field term. The equilibrium state is first discussed. The dispersion relation for collective oscillations (monopole, quadrupole, dipole modes) is then derived. In particular, our treatment gives the frequency of the monopole mode in an isotropic and harmonic trap in the presence of mean field in all dimensions.

The dynamics of Bose-Einstein condensates (BEC) of dilute atomic gases are described by the Gross-Pitaevskii equation ([1] and references therein). The main feature of this equation is the mean-field term arising from interaction between particles. Most of BEC experiments are in the so-called Thomas-Fermi regime for which the interaction energy dominates the kinetic energy, resulting in an inverted parabola density shape of the condensate. However, the mean-field term is not found only in a bosonic gas well below its critical temperature T_c. Such a contribution from collisions also exists above T_c, and is even magnified by a kind of Hanbury-Brown and Twiss factor. Up to now most BEC experiments have been performed in the collisionless regime (when the mean free collision rate is small relative to the trap frequency) and with a negligible contribution of the mean-field of the non-condensed atoms.

In order to specify the role of dimensionality, we introduce the adimensional parameter $\zeta = gn/k_BT$ *i.e.* the ratio between the mean field energy and the thermal energy. One readily establishes that $\zeta_{3D} \sim (na^3)^{1/3}(n\lambda_{dB}^3)^{2/3}$, where $\lambda_{dB} = h(2\pi mk_BT)^{-1/2}$ is the de Broglie wavelength and a the s-wave scattering length. Consequently for a dilute bosonic gas above the critical temperature: $\zeta_{3D} \ll 1$ and results presented in this paper are valid as corrections. In the weakly interacting limit, the 2D quantity ζ_{2D} is only logarithmically small with respect to λ_{dB}^2 and mean field energy can be comparable to thermal energy above the quantum transition temperature (Kostherlitz-Thouless [2]). The quantity ζ_{1D} is of the order of $(n\lambda_{dB})^2(nl_c)^{-2}$ where the correlation length $l_c = \hbar/\sqrt{mgn}$. Classical description and mean field can be used up to the regime where $\lambda_{dB} \sim 1/n \sim l_c$ so up to $\zeta_{1D} \sim 1$. In the regime where l_c is much smaller than the mean interparticle separation the gas acquires Fermi properties and is called a gas of impenetrable bosons or Tonk gas [3].

Finally, by reduction of the dimensionality the role of the mean field even above the critical temperature of a quantum transition can be important. Experiments performed on microchip offer the possibility to investigate low dimensional regime [4,5] as well as experience with dipolar trap or/and magnetic trap [6].

In this paper, our aim is to extend the traditional treatment of the bosonic gas above the critical temperature by taking into account the effect of particle interactions. The method consists in including the classical mean field term, also known as the Vlasov contribution, within the Boltzmann equation. So far, collective oscillations of a bosonic gas above the critical temperature have been investigated without mean field contribution in the hydrodynamic regime in Ref. [7,8], and an interpolation formula from the collisionless up to the hydrodynamic regime has been proposed in Ref. [9,10].

In Sec. I, we briefly recall the general framework that describes how the mean field is taken into account in the Boltzmann equation. The stationary solution is discussed in Sec. II. In Sec. III, we derive equations of the low energy collective excitation of a Bose gas for positive and negative scattering length by means of a scaling ansatz. We report an interpolation formula from the collisionless gas to the interaction dominated thermal gas (Vlasov gas) in the absence of dissipation.

I. FORMULATION

In traditional BEC experiments, the bosonic gas above the critical temperature is well-described by the classical Boltzmann equation [11], or the Uhlenbeck-Boltzmann equation [12] if experiments are sufficiently accurate to measure deviation from the classical distribution. In the following, we present the extension of this equation when the mean-field is taken into account.

We consider an ensemble of harmonically trapped thermal atoms that evolves according to the Boltzmann-Vlasov kinetic equation [11,13]:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} - \frac{\partial U}{\partial \mathbf{r}}\cdot\frac{\partial f}{\partial \mathbf{v}} - 2\frac{g}{m}\frac{\partial n}{\partial \mathbf{r}}\cdot\frac{\partial f}{\partial \mathbf{v}} = I_{\text{coll}} \qquad (1)$$

where $f(\mathbf{r},\mathbf{v},t)$ is the single particle phase space distribution function, $U = \sum_i \omega_i^2 r_i^2/2$ the confining potential, $n = \int f d^D v$ the density, where g is the mean field strength in D dimensions. In 3D, $g = 4\pi\hbar^2 a/m$ is the

strength of the pseudopotential replacing the true two-body potential at low energies , with the s-wave scattering length a. $I_{\rm coll}$ is the collisional integral that describes relaxation processes. Note that $I_{\rm coll} = 0$ in 1D because of conserved quantities. The Vlasov term (last term of the l.h.s. of (1)) is a Hartree-Fock mean-field term [14] and is even magnified for point-like interactions by a factor of two with respect to the condensate for the same density. Indeed, for non-condensed cloud both the Hartree and the Fock terms contribute, whereas only the Hartree term contributes for the condensate.

The kinetic equation (1) is valid for $k_BT \gg \hbar\omega$ where ω is the typical trap frequency and for $a \ll a_0$ where $a_0 = (\hbar/(m\omega))^{1/2}$ is the oscillator quantum length. One has to check that the s-wave approximation is valid. In 3D it requires that $a \ll \lambda_{dB}$.

II. EQUILIBRIUM STATE

At equilibrium, the Eq. (1) reads:

$$\sum_{i=1}^{D}\left(v_i\frac{\partial f_0}{\partial r_i} - \omega_i^2 r_i\frac{\partial f_0}{\partial v_i} - \frac{2g}{m}\frac{\partial n_0}{\partial r_i}.\frac{\partial f_0}{\partial v_i}\right) = 0. \qquad (2)$$

By multiplying Eq. (2) by $v_j r_j$ and integrating over space and velocity, we deduce the average size $\langle r_j^2\rangle$ along the j axis [9]:

$$\omega_j^2\langle r_j^2\rangle - \langle v_j^2\rangle - \frac{g}{mN}\int n_0^2 d^D r = 0. \qquad (3)$$

As expected, repulsive interactions ($g > 0$) favor a reduction of the density from the free particle situation. The opposite behavior is obtained in case of attractive interactions ($g < 0$). One can extract the shape of the density by searching for a factorized solution of (2) of the form: $f_0(\mathbf{r},\mathbf{v}) = n_v(\mathbf{v})n_0(\mathbf{r})$. One finds a Gaussian spherical distribution for the velocity. The density distribution is a solution of the following equation:

$$\kappa \ln n_0 + 2gn_0/m = \mu - \sum_j \omega_j^2 r_j^2/2, \qquad (4)$$

where $\kappa = \int v_j^2 n_v d^D v = k_BT/m$. In two limiting cases, the solution has a simple form. For $g = 0$, we find the gaussian shape as expected for an harmonic confinement without the Vlasov term. On the contrary, in the limit in which the interparticle interactions dominate and are repulsive, the shape of the cloud is determined by a balance between the harmonic oscillator and interactions energy resulting in approximately an inverted parabola shape for repulsive interactions. This is the same shape as found for a harmonically trapped BEC in the Thomas-Fermi regime [1], since in this case the mean-field term also dominates. Note that this last result can also be shown in the classical hydrodynamic regime [8] under the same conditions. Strictly speaking this situation can be reached only for low dimensional system. For intermediate g, Eq. (4) gives the proper interpolation between the gaussian and the Thomas-Fermi shape.

For $g < 0$, the density distribution is sharpened with respect to the free gaussian one. Actually, by increasing the number of atoms, the spatial extent of the distribution is reduced. If attractive forces overwhelm the kinetic energy, the cloud should collapse. One can work out the criterium for such a collapse in 3D by means of a gaussian ansatz [15] and finds $a_c = 33a_0N^{-1}(a_0/\lambda_{dB})^5$. However this result is out of the range of validity of the classical approximation.

III. COLLECTIVE OSCILLATIONS OF A COLLISIONLESS GAS

In this section, we investigate the collective oscillations of a Vlasov gas *i.e.* in the absence of the dissipative term ($I_{\rm coll}$) but with the mean field contribution.

A. Scaling ansatz method

We study those modes by means of the scaling factor method [8,16–19] in D dimensions. We recall that in this method the proper shape of the cloud does not enter directly in the equations. This is the reason why the solutions are equally valid for a Bose gas just above the critical temperature as for a classical gas. We make the following ansatz for the non equilibrium distribution function: $f(\mathbf{r},\mathbf{v},t) = f_0(\mathbf{R}(t),\mathbf{V}(t))$ with $R_i = r_i/\lambda_i$ and $V_i = \lambda_i v_i - \dot{\lambda}_i r_i$. The dependence in t is contained in the free parameters λ_i. By substituing this ansatz into Eq. (1), we find:

$$\sum_i\left\{\frac{V_i}{\lambda_i^2}\frac{\partial f_0}{\partial R_i} - \lambda_i R_i(\ddot{\lambda}_i + \omega_i^2\lambda_i)\frac{\partial f_0}{\partial V_i} - \frac{2g}{\Pi_j\lambda_j}\frac{\partial n_0}{\partial R_i}\frac{\partial f_0}{\partial V_i}\right\} \simeq 0. \qquad (5)$$

This equation can be combined with Eq. (2) taken in the phase space point ($\mathbf{r} = \mathbf{R}, \mathbf{v} = \mathbf{V}$) in order to replace the last term of (5) by a linear superposition of $\partial f_0/\partial R_i$ and $\partial f_0/\partial V_i$. We finally obtain:

$$\sum_i\left\{\left(\frac{V_i}{\lambda_i^2} - \frac{V_i}{\Pi_j\lambda_j}\right)\frac{\partial f_0}{\partial R_i} - \lambda_i R_i\left(\ddot{\lambda}_i + \omega_i^2\lambda_i - \frac{\omega_i^2}{\lambda_i\Pi_j\lambda_j}\right)\frac{\partial f_0}{\partial V_i}\right\} = 0. \qquad (6)$$

This equation provides the constraints on our ansatz. The first average moment [9] of R_iV_i, namely $\int R_iV_i[Eq.(6)]d^DRd^DV/N$, leads to a set of Newton-like second order ordinary differential equations:

$$\ddot{\lambda}_i + \omega_i^2\lambda_i - \frac{\omega_i^2}{\lambda_i^3} + \omega_i^2\xi\left(\frac{1}{\lambda_i^3} - \frac{1}{\lambda_i\Pi_j\lambda_j}\right) = 0 \qquad (7)$$

with $\xi = g\langle n_0\rangle/(g\langle n_0\rangle + k_BT)$. To find the excitation frequencies of the modes, one must linearize around the equilibrium value: $\lambda_i = 1$.

B. Monopole mode

Consider the case of an isotropic harmonic confinement ($\omega_i = \omega_0$). In this case the scaling ansatz is in the kernel of the collision integral and provides a solution of (1) valid also in the collisional regime. An exact solution of this mode was first reported in [20] for the classical Boltzmann equation without mean field.

The small amplitude expansion of (7) gives the frequency of the monopole mode (also called the breathing mode): $\omega_0\sqrt{4+\xi(D-2)}$, for all dimensions and in presence of the total effects of collisions (mean field term and dissipation *via* $I_{\rm coll}$). In 3D this frequency ranges from $2\omega_0$ in the absence of the mean field term up to $\sqrt{5}\omega_0$ when the mean field dominates, in the latter case one obtains the same result as expected for Bose-Einstein condensate in the Thomas-Fermi limit [25]. In 2D we find $2\omega_0$ for the monopole mode a result independent of the mean field, a special feature of 2D already investigated in Ref. [22]. In this case, the scaling ansatz provides an exact solution of (1). In 1D, the monopole frequency ranges from $2\omega_0$ down to $\sqrt{3}\omega_0$ when the mean field dominates. Note that this latter result gives exactly the same frequency as the one of the monopole mode in trapped ions [23]. For ions the force originates from the coulomb interaction, this long range force is well described by mean field and this is probably the reason why we recover the same result.

C. Quadrupolar mode

In 2D and 3D, Eq. (7) provides the mean field contribution to quadrupolar collective oscillations for a collisionless gas.

In 2D, Eq. (7) gives two coupled equations for λ_1 and λ_2, which, after linearization, yield the dispersion relation

$$\omega^2 = \frac{1}{2}\Big[(4-\xi)(\omega_x^2+\omega_y^2) \pm((4-\xi)^2(\omega_x^2+\omega_y^2) - 32\omega_x^2\omega_y^2)^{1/2}\Big]. \qquad (8)$$

For $\xi = 0$, we recover the single particle excitation frequency of the cloud : $\omega = 2\omega_i$ for each spatial direction. In the limit $\xi = 1$, this relation can be derived from a purely hydrodynamic approach [7,8] by taking into account the mean field contribution in the same limit. Formula (8) provides also finite temperature corrections to this regime and the proper interpolation in between those two limiting cases.

For a cylindrical 3D harmonic trap (we denote $\beta = \omega_z/\omega_\perp$) we find the eigenfrequencies of mode $M = 0$ (coupling between quadrupole and monopole modes):

$$\omega^2 = \frac{1}{2}\Big[4+4\beta^2-\xi \pm([4+4\beta^2-\xi]^2 + 8\beta^2[-8+2\xi+\xi^2])^{1/2}\Big], \qquad (9)$$

and the frequency of the quadrupole mode with azimuthal quantum number M=2: $\omega^2/\omega_\perp^2 = 2(2-\xi)$. The limit $\xi \sim 1$ gives the formulas derived by Stringari [25] for the low energy excitation spectrum of a BEC in the limit for which the energy of interaction predominates over the kinetic energy. Low energetical collective excitations (monopole, quadrupole) spectrum of a BEC are rather a proof of mean field dominated physics than a direct proof of superfluidity in the Landau sense [24]. As already pointed out, the validity of our calculation in 3D is only perturbative ($\xi \ll 1$). For an isotropic 3D trap ($\omega_i = \omega_0$), the oscillation frequency split into the monopole mode with a frequency $\omega_M \simeq 2\omega_0(1+\xi/8)$ and the quadrupole mode with a frequency $\omega_Q \simeq 2\omega_0(1-\xi)$ as soon as we take into account the mean field.

D. Dipolar mode

The dipolar mode which corresponds to the rigid motion of the density profile is not affected by the mean field. This can be shown by searching for a solution of the form $f(\mathbf{r},\mathbf{v},t) = f_0(\mathbf{R}(t),\mathbf{V}(t))$ with $R_i = r_i - \eta_i$ and $V_i = v_i - \dot{\eta}_i$. Each component η_i is time dependent. Following the same procedure, we readily establish the equation of motion for η by taking the average value of V_i: $\ddot{\eta}_i + \omega_i^2\eta_i = 0$. We recover the fact that Kohn modes do not depend on interactions. This last result is naturally unchanged if we take into account the collisional integral contribution.

E. Discussion

Note that $I_{\rm coll} = 0$ is strictly speaking only applicable for a collisionless gas ($a \to 0$) and to the hydrodynamic regime (collision rate $\gg$ trap frequencies). In between, $I_{\rm coll}$ is negligible with respect to the Vlasov term if $a \ll \tilde{a}$, where $\tilde{a}$ is a critical value for the scattering length. The collision integral is of the same order of magnitude as the Vlasov contribution when $g = \sigma v^2\ell$, where $\sigma = 8\pi a^2$ is the elastic cross section, v is a typical thermal velocity and ℓ a typical size. We take $\ell \sim v/\omega$ and find $\tilde{a} = 0.03\lambda_{dB}(\lambda_{dB}/a_0)^2$. Just above the critical temperature $\tilde{a}$ is of the order of the scattering length for the experiment of Ref. [4] performed on a microchip. For the

metastable helium experiment [26] $\tilde{a}$ is slightly smaller than the scattering length. In these experiments even if they are not in the Thomas-Fermi regime, one can no longer ignore mean field effects. The interpolation parameter in Eq. (7) is the ratio between the mean field and the thermal energy: $\zeta = gn/k_B T$. In many BEC experiments, $\zeta_{max} < 10^{-4}$ which clearly justifies that it is neglected. However, in Ref. [4] this ratio is of the order of $\zeta_{max} \sim 10\%$ and could be increased by a stronger longitudinal confinement. In [26], this ratio is of the order of $\zeta_{max} \sim 20\pm10\%$ as a consequence of the huge value of the scattering length and the high density of the sample.

IV. CONCLUSION

Mean field effects for a bosonic gas above quantum transition play an increasing role as the dimension is reduced. This paper deals with the contribution of the mean field to the low energetical collective modes of such a gas. We derive, even for a collisional gas, the frequency of the monopole mode for an isotropic and harmonic confinement. Note that results derived in this article hold also for two-components Fermi-system as soon as the mean field play a role.

The mean field contribution could be seen directly on the equilibrium shape of the gas *in situ*. Time-of-flight measurement may give relevence to a direct observation of mean field contribution [27].

Further experiments on microchip [5] should allow the reduction of dimensionality. Dipolar trap can also be a good tool [6]. Those techniques should help for the observation of classical mean field above a quantum transition.

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[1] F. Dalfovo, S. Giorgini, L. Pitaevskii and S. Stringari Rev. Mod. Phys. **71**, 463 (1999).

[2] J.M. Kosterlitz and D.J. Thouless, J. Phys. C **6**, 1181 (1973); J.M. Kosterlitz J. Phys. C **7**, 1046 (1974).

[3] E.H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963); E.H. Lieb, Phys. Rev. 130, 1616 (1963); M. Girardeau, J. Math. Phys. (N.Y.) **1**, 516 (1960); L. Tonks, Phys. Rev. **50**, 955 (1936).

[4] W. Hansel, P. Hommelhoff, T.W. Hansch and J. Reichel, Nature **413**, 498 (2001).

[5] H. Ott, J. Fortagh, G. Schlotterbeck, A. Grossmann and C. Zimmermann, Phys. Rev. Lett. **87**, 230401 (2001).

[6] V. Vuletic, C. Chin, A.J. Kerman and S. Chu, Phys. Rev. Lett. **81**, 5768 (1998); V. Vuletic, A.J. Kerman, C. Chin and S. Chu, Phys. Rev. Lett. **82**, 1406 (1999);I. Bouchoule, H. Perrin, A. Kuhn, M. Morinaga and C. Salomon, Phys. Rev. A **59**, R8 (1999);M. Morinaga, I. Bouchoule, J.-C. Karam and C. Salomon, Phys. Rev. Lett. **83**, 4037 (1999); A Gorlitz, J M Vogels, A E Leanhardt *et. al.* Phys. Rev. Lett. **87**, 130402 (2001); F Schreck, L Khaykovich, K L Corwin *et. al.* Phys. Rev. Lett. **87**, 080403 (2001).

[7] A. Griffin, Wen-Chin Wu, and S. Stringari, *Phys. Rev. Lett.* **78**, 1838 (1997).

[8] Y. Kagan, E.L. Surkov and G.V. Shlyapnikov, Phys. Rev. A **55**, R18 (1997).

[9] D. Guéry-Odelin, F. Zambelli, J. Dalibard and S. Stringari, Phys. Rev. A **60**, 4851 (1999).

[10] U. Al Khawaja, C. Pethick, H. Smith, Jour. of Low Temp. Phys. **118**, 127 (2000).

[11] K. Huang, *Statistical Mechanics*, (J. Wiley, New York, 1987), 2nd ed.

[12] L.P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W.A. Benjamin, New York, 1962).

[13] H. Kreuzer, *Non equilibrium Thermodynamics and its statistical Foundations* (Clarendon Press, 1981).

[14] Y. Castin, Session LXXII , in: R. Kaiser, C. Westbrook, F. David (Eds.), Coherent atomic matter waves "Les Houches", EDP Sciences; Springer-Verlag 2001.

[15] G. Baym and C.J. Pethick, Phys. Rev. Lett. **76**, 6 (1996).

[16] Y. Castin and R. Dum, Phys. Rev. Lett. **77**, 5315 (1996).

[17] V.M. Perez-Garcia, H. Michinel, J.I. Cirac, M. Lewenstein and P. Zoller, Phys. Rev. Lett. **77**, 5320 (1996).

[18] K.G. Singh and D.S. Rokhsar, Phys. Rev. Lett. **77**, 1667 (1996).

[19] M.J. Bijlsma and H.T.C. Stoof, Phys. Rev. A **60**, 3973 (1999).

[20] L. Boltzmann, in *Wissebschaftliche Abhandlungen*, edited by F. Hasenorl (J.A. Barth, Leipzig, 1909), Vol II, p. 83.

[21] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996).

[22] L.P. Pitaevskii and A. Rosch, Phys. Rev. A **55**, R853 (1997).

[23] D.F.V. James, Appl. Phys. B **66**, 181 (1998).

[24] L.D. Landau, J. Phys. **5**, 71 (1941).

[25] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996).

[26] F. Pereira Dos Santos, J. Léonard, J. Wang, C.J. Barrelet, F. Perales, E. Rasel, C.S. Unnikrishnan, M. Leduc and C. Cohen-Tannoudji, Phys. Rev. Lett. **86**, 3459 (2001).

[27] One may argue that the asymptotic signal of a time of flight measurement should reflect the isotropy of the velocity distribution. Nevertheless, anisotropic forces induced by collisions can break this isotropy for strong interacting regime, see Ref. [8] and H. Wu and E. Arimondo, Phys. Rev. A **58**, 3822 (1998).